ECO 328: A Primer on Linear Algebra and Multivariate Calculus

This course relies on mathematical tools above those covered in the prerequisites. We this begin with a quick primer on this material. We will focus only on the aspects of linear algebra that we will use heavily in the class. The treatment that follows is operational rather than theoretical.

A *matrix* is simply a collection of individual elements. For example, the matrix A may be:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{1}$$

Here, $a_{1,1}$ is the element from the first row and first column of A. , $a_{2,1}$ is the element from the second row and first column of A...

A matrix is said to be MxN if it consists of M rows and N columns. A, for example, is $2x2$.

A *vector* is a matrix consisting of either one row or one column and a *scalar* is a $1x1$ matrix.

Addition

Adding matrices is very simple. We simply sum the matching elements for each matrix. For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \tag{2}$$

In order for matrices to be added, they must be of the same dimension (same number of rows and columns). If not, they are not conformable and their sum is not defined.

Multiplication

To multiply matrices, we do not simply take the product of the corresponding elements:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \neq \begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix} \tag{3}$$

Considering $A * B = C$, the first row of A multiplies the first column of B to obtain $c_{1,1}$.

$$c_{1,1} = a_{1,1} * b_{1,1} + a_{1,2} * b_{2,1} + \ldots a_{1,m} * b_{m,1} \tag{4}$$

The second row of A then multiplies the first column of B to obtain $c_{2,1}$.

$$c_{2,1} = a_{2,1} * b_{1,1} + a_{2,2} * b_{2,1} + \ldots a_{2,m} * b_{m,1} \tag{5}$$

The first row of A multiplies the second column of B to obtain $c_{1,2}$.

$$c_{1,2} = a_{1,1} * b_{1,2} + a_{1,2} * b_{2,2} + \ldots a_{1,m} * b_{m,2} \tag{6}$$

It thus follows that:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \tag{7}$$

For matrix multiplication to work, the number of rows of A must equal the number of columns of B. Otherwise their product is not defined. If A is MxN, and B is NxP, then their product is MxP.

Note that, in general, $A * B \neq B * A$.

Mutiplying a matrix by a scalar is different. The following example shows how this is done:

$$\kappa \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \kappa 1 & \kappa 2 \\ \kappa 3 & \kappa 4 \end{bmatrix} \tag{8}$$

Inversion

The *identity matrix* is the matrix equivalent of the number one.$I(3)$ the$3x3$ identity matrix is:

$$I(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

For $I(n)$, the identity matrix consist of ones on all n diagonals and zero for all off-diagonal elements. The inverse of A, denoted as A^{-1}, has the property that:

$$A * A^{-1} = I = A^{-1} * A \tag{10}$$

It is aso the case that:

$$A * I = A = I * A \tag{11}$$

For large matrices, there is no simple formula for computing an inverse. For a two by two matrix, however, a simple formula does exist:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix} \tag{12}$$

The inverse of a matrix is only defined for square matrices (those with the same number of rows and columns). A square matrix is not always invertible, an inverse exists if and only if the matrix's *determinate* is non-zero. For a $2x2$ matrix, the determinate equals $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

A $2x2$ matrix is non-invertible if $a_{1,1}a_{2,2} = a_{1,2}a_{2,1}$. This occurs if one row (or column) is a multiple of the other row (or column). For example, the following matrix is not invertible:

$$\begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$$

Transpose

The transpose of a matrix is replaces its columns with its rows and vice-versa. For example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{13}$$

$$A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \tag{14}$$

Eigenvalues and Eigenvectors

The *eigenvalues* of square matrices will be useful throughout the term. The eigenvalues (denoted as the vector λ) is obtained through the *characteristic equation*:

$$Det\,(A - \lambda I) = 0 \tag{15}$$

For each eigenvector, λ_i, it is possible to obtain an eigenvector, x using the following formula:

$$(A - \lambda_i I)x = 0 \tag{16}$$

Do not be alarmed if you do not see the usefulness of eigenvalues. It will become clearer when we apply this background to macroeconomic material later in the term.

An Example

Consider the following matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To obtain this matrix's eigenvalues, we set the determinate of the following matrix equal to zero:

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \tag{17}$$

which using the formula for the determinate of a $2x2$ matrix sets:

$$(1-\lambda)(4-\lambda)-6=0; \tag{18}$$

The solution to (18) is a quadratic it may be rewritten as:

$$\lambda^2-5\lambda-2=0 \tag{19}$$

Using the quadratic formula:

$$\lambda_i = \frac{5 \pm -\sqrt{25+8}}{2} = -0.37, 5.37 \tag{20}$$

To find eigenvectors, consider $\lambda_i = 5.37$:

$$\begin{bmatrix} 1-5.37 & 2 \\ 3 & 4-5.37 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{21}$$

which yields:

$$\begin{bmatrix} -4.37x_1+2x_2 \\ 3x_1-1.37x_2 \end{bmatrix} = 0 \tag{22}$$

Both equations reduce to $x_2 = 2.19x_1$. Any non-zero vector that satisfies this relationship is thus an eigenvector. The solution may thus be written as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.19 \end{bmatrix} \tag{23}$$

Eigenvectors are not unique. Multiplying an eigenvector by any non-zero scalar (zero vectors are excluded from being eigenvectors) also returns a valid eigenvector.

Application #1: Representing a system of equations:

Serious macroeconomic models almost always consist of a set of equations. Consider the following example:

$$\pi_t + \alpha y_t + \beta i_t = u_t \tag{24}$$

$$\pi_t + \gamma y_t + \omega i_t = g_t \tag{25}$$

$$\pi_t + \chi y_t + \kappa i_t = e_t \tag{26}$$

Where π_t is inflation y_t is output, i_t is the interest rate, and the variables on the right hand side are exogenous shocks. Solving this model consists of representing the three endogenous variables as functions of exogenous shocks and parameters. Because there are three equations and three endogenous variables, the system is well-specified. It is possible through tedious substitution, to do this using (24)-(26). Our life is much easier, however if we represent the model using matrices:

$$Ax_t = z_t \tag{27}$$

$$\begin{bmatrix} 1 & \alpha & \beta \\ 1 & \gamma & \omega \\ 1 & \chi & \tau \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \\ ei_t \end{bmatrix} = \begin{bmatrix} u_t \\ g_t \\ e_t \end{bmatrix} \tag{28}$$

Notice that by multiplying the matrices on the left hand side we return (24)-(26). Using linear algebra we can work with a single equation instead of three. Solving the model is easy:

$$x_t = A^{-1} z_t \tag{29}$$

Application #2: Regression Analysis

Linear algebra allows us to more easily apply econometrics. Suppose that some vector, y is our dependent variable and that some matrix, X is the set of our independent variables which most likely includes a column of ones so that the model incudes a constant. It can be shown that $\hat{\beta}$, the OLS estimator is simply:

$$\hat{\beta} = (X^{‘}X)^{-1}X^{‘}y \tag{30}$$

There is thus no need to treat univariate and multivariate regression analysis as spearate topics if we use matrices.

Application #3: Eigendecomposition

Suppose that we have the following model where D is $3x3$:

$$x_t = Dz_t \tag{31}$$

It will often be useful to *decompose* the matrix D using its eigenvalues and eigenvectors. Define Λ as a matrix of eigenvalues on the diagonals and zeros on all off-diagonal elements:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{32}$$

Define S as a matrix of $D's$ eigenvectors. The first column of S is the eigenvector corresponding to λ_1, the second column of S is the eigenvector corresponding to λ_2, and the third column of S is the eigenvector corresponding to λ_3. It is important that the eigenvectors be in the correct order. If S is invertible, then it is true that:

$$D = S\Lambda S^{-1} \tag{33}$$

We can then define $q_t = S^{-1}x_t$. Pre-multiply both sides of (31) by S^{-1}:

$$S^{-1}x_t = S^{-1}S\Lambda S^{-1}z_t \tag{34}$$

$$qz_t = \Lambda S^{-1}z_t \tag{35}$$

Equation (35) is just another way of writing (31). There is no reason why its usefulness should be apparent to you at this time. But this technique will be helpful later in the class.

Partial Derivatives

Suppose that a function depends of multiple variables:

$$Y = AK^{\frac{1}{3}}L^{\frac{2}{3}} \tag{36}$$

Now suppose that we want to see how changing a RHS variable affects the LHS variable. Were there just a single RHS variable, we would just take an ordinary derivative. In a multivariate setting, we can do similar. By holding the other variables constant, we can take a partial derivative.

$$\frac{\partial Y}{\partial K} = \frac{1}{3}AK^{\frac{-2}{3}}L^{\frac{2}{3}} \tag{37}$$

There are two differences between a partial and ordinary derivative. First, a partial derivative only equals the instantaneous rate of change if the other variables are held constant. This may be valid if the RHS consists only of exogenous variables. Second, it uses the more fashionable ∂ symbol.

Likewise, we can take the partial derivative of Y with respect to L.

$$\frac{\partial Y}{\partial L} = \frac{1}{3}AK^{\frac{1}{3}}L^{\frac{-1}{3}} \tag{38}$$