The Solow Model[1]

Some General Comments on Growth

Before we begin the Solow Model, lets consider a few basics on growth, most of which you have seen in your previous coursework:

1. Our goal, and this will be equally true with business cycle models later in the course, is to develop models that i) rely on plausible microeconomic based assumptions, and ii) yield predictions that are sufficiently consistent with the data. In analyzing growth, we will begin with the Solow Model. This model is best seen as a starting point rather than a complete success in the context of these goals.

2. Growth is the study of long-run macroeconomics. We are making no attempt to explain business cycles. Rather we are attempting to answer questions such as; i) why are some countries 20-30 times wealthier than others and ii) why have some countries exhibited sustained growth since about 1850 which has dramatically elevated living standards.

The Solow Model (1956)

This is the standard model to begin studying growth and is included in macroeconomics courses at all levels from Principles to Ph.D. It has a strange history. It did not make a huge impact upon its publication but was rediscovered in the 1980s. These notes follow portions of Chapter 1 of Romer.

As always, we begin by making the model's key assumptions explicit:

Core Assumptions

a. The production function is:

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

where $Y(t)$ is output, $K(t)$ is physical capital, $L(t)$ is labor, and $A(t)$ is total factor productivity (hereafter TFP), a residual that captures all other inputs besides capital and labor. TFP includes, but is not limited to, technology and human capital.

[1] These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

Notice that TFP multiplies labor and not capital. This is known as "labor augmentation." All variables are measured in continuous time.

b. TFP and labor evolve according to:

$$\dot{A}(t) = gA(t) \tag{2}$$

$$\dot{L}(t) = nL(t) \tag{3}$$

where g is the exogenous rate of TFP growth and n is the exogenous rate of population growth. Both assumptions are controversial. Endogenizing g is a major goal of the Endogenous s Growth literature and endogenizing n is part of the smaller endogenous fertility literature.

Note that both (2) and (3) are first order differential equations like those examined in the previous mathematical primer. It thus follows that their solutions follow:

$$A(t) = A(0)e^{gt} \tag{4}$$

$$L(t) = L(0)e^{nt} \tag{5}$$

c. The production function exhibits constant returns to scale with respect to $K(t)$ and $A(t)L(t)$. Formally:

$$F(cK(t), cA(t)L(t)) = cF(K(t), A(t)L(t)) = cY(t) \tag{6}$$

where $c > 0$. This assumption implies that if we double both capital and labor, then we double output.

Because we are assuming constant returns to scale, we may re-write- the production function in its *intensive form*. Define $x(t) = \frac{X(t)}{A(t)(L(t)}$. This definition changes the interpretation of any variable to "units per effective unit of labor." Although we do not care about this variable directly, it will be useful for solving the model.

Set $c = \frac{1}{A(t)L(t)}$. From (6), this yields:

$$y(t) = \frac{Y(t)}{A(t)L(t)} = F(k(t), 1) = f(k) \tag{7}$$

Equation (7) is the intensive production function. Note that it makes some notational changes, including dropping the (t).

d. The intensive production function satisfies the following conditions:

$$f(0) = 0 \tag{8}$$

$$f^{'}(k) > 0 \tag{9}$$

$$f^{''}(k) < 0 \tag{10}$$

and the *Inada Conditions*:

$$lim_{k \to \infty} f^{'}(k) = 0 \tag{11}$$

$$lim_{k \to 0} f^{'}(k) = \infty \tag{12}$$

Collectively, (8)-(12) yield the following graph:

Graph: The Intensive Production Function

e. Households save a constant and exogenous fraction of their output s. This is the great weakness of the Solow Model. This assumption is *ad-hoc* and is not based on microeconomic utility maximization. As a result, it is difficult to talk about welfare, and hence optimal allocations.

f. The economy is closed (no trade, and no government). It is therefore the case that investment equals savings, $I(t) = S(t)$. Recall that investment is the creation of new capital.

g. A constant and exogenous fraction of capital, δ, depreciates. because this assumption concerns technology and not choices, it is not controversial. We further assume that $n+g+\delta > 0$.

The Cobb-Douglas Form

Often, we will assume the following, more specific, production function:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \tag{13}$$

It is direct to verify that this functional form satisfies our earlier assumptions. It also has empirical support. Suppose that markets are competitive. If so, then we would expect that the marginal product of labor will equal the wage:

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A^{1-\alpha}L^{-\alpha} \tag{14}$$

Note that Y is total income. The total amount of wages collected is wL. It then follows that labor's share of income is:

$$\frac{wL}{Y} = 1 - \alpha \tag{15}$$

and similar steps can show that capital's share is α. Empirical evidence shows that capital's share has been remarkably stable around $\alpha = \frac{1}{3}$. This is often taken as evidence in favor of the Cobb-Douglas functional form.

The dynamics of k

It follows from assumptions $e - g$ that:

$$\dot{K}_t = sY(t) - \delta K(t) \tag{16}$$

This is the *capital accumulation equation*. The first term on the right hand side is investment; the creation of new capital. It is simply output multiplied by the fraction that household's exogenously save. the second term on the right hand side is depreciation, the amount of capital lost. Recall that $\dot{k}(t) = \frac{K(t)}{A(t)L(t)}$, we can use the chain rule to re-state (16) in terms of k:

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}[A(t)\dot{L}(t) + L(t)\dot{A}(t)] = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{L}(t)}{L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{A}(t)}{A(t)} \tag{17}$$

which can then be combined with (2), (3), and (15) to yield:

$$\dot{k}(t) = \frac{sY(t) - \delta K(t)}{A(t)L(t)} - (n+g)k(t) = s\frac{Y(t)}{A(t)L(t)} - (n+g+\delta)k(t) \tag{18}$$

Finally, use (7) to eliminate $Y(t)$:

$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t) \tag{19}$$

The Solow Model thus reduces to a single difference equation. All other variables are simple functions of k.

Steady States and Stability

The steady state is defined so that if the model's variables equal their steady state values, they will then remain there indefinitely. Formally, the steady state, k^* is defined so that $\dot{k}^* = 0$. From (19) the amount of new capital per effective unit of labor, $sf(k(t))$ equals the amount of capital lost due to depreciation, $(n+g+\delta)k(t)$. On a graph, a steady state occurs where these two functions intersect:

Graph: The Solow Model:

The Solow Model has two steady states:

1. The Binghamton, New York steady state. Here $k = 0$. Using our assumption from (8), $f(0) = 0$. Because nothing is produced, nothing is saved or invested and no capital can be created. Likewise, no capital can be lost to depreciation.

This steady state is not very interesting for two reasons. First, it is clearly empirically implausible as even the world's poorest economies have positive output. Second, this steady state is unstable. Suppose that the economy is initially at $k = \epsilon$, where ϵ is arbitrarily close to zero. Imagine that k equals one -trillionth of one cent. Using (12), $f^{'}(k)$ will be very large and $\dot{k}$ will thus be as well. Thus if the model is ever close to the zero steady state, the capital stock will begin to grow.

This same result can be seen in the graph. For low enough values of k, the investment function lies above the depreciation function. More capital is created than lost and net capital creation is thus positive.

2. Using (11) and (12), the slope of $sf(k)$ is initially very large and then approaches zero as capital goes to infinity. Because $(n + h + \delta)$ is positive and constant, it must be the case that the two functions intersect for some positive value of capital, k^*. This is the interior steady state.

It is clear from our graph, that this steady state is stable. Note that if the economy lies to its left, then investment is greater than depreciation. If, however, the economy lies to the right of k^*, the depreciation is greater than investment.

At k^*, capital per unit of effective labor is constant. We do not really care about this variable, however, We are more interested in how capital and output per worker are changing. Denote capital per worker as $\hat{k}$. It is clear that:

$$\hat{k(t)} = A(t)k(t) \tag{20}$$

We can now use logs to calculate the growth rate of capital per worker. The growth rate of a variable equals the first derivative of its log. So it is helpful to take logs of (20):

$$ln(\hat{k}) = ln(A(t)) + ln(k(t)) \tag{21}$$

where the derivation of (21) uses the property that $ln(XZ) = ln(X) + ln(Z)$. Equation (21) shows that the growth rate of capital per worker must be equal to the growth rate of TFP plus the growth rate of capital per effective units of labor. By definition, the growth rate of k is zero at a steady state. So at a steady state, the growth rate of capital per worker equals g, TFP's exogenous growth rate. Convergence toward a steady state therefore represents changes to this growth rate above or below g. In the steady state, however, TFP growth is the only source of economic growth.

This last result helps explain why poorer countries, such as China or India, should exhibit higher growth rates than developed economies like the United States or Japan. The latter have only TFP growth. The former have an additional source of growth as their economies converge toward the steady state. this assumes that developed economies are near a common steady state while developing economies are further away.

Using our assumptions, we can say more about the process of convergence. The growth rate of capital per unit of effective labor is proportional to the gap between the investment and depreciation functions:

1. Near the zero steady state, this gap is very small, $\dot{k}$ is thus positive, but close to zero.

2. Because (12) assumes that the investment function is very steep when k is small, this gap is growing for low values of k.

3. Because these functions intersect at the positive steady state, this gap must peak, shrink, and eventually reach zero at the positive steady state. Assumption (10) ensures that $\dot{k}$ is smooth.

4. Beyond the positive steady state, $\dot{k}$ becomes increasingly negative.

Graph: phase diagram for $\dot{k}$

Effects of changing s

Increasing s increases the level of investment in the economy. It is clear from a graph that such a change increases k^*.

Graph: Increasing s

Less obvious is the effect on steady state consumption, c^*. Increasing s increases output, but reduces the share of output that is available for consumption. These two effects work in the opposite direction. We now solve for the value of s that maximizes c^*, known as the golden rate of saving. Note that:

$$c^* = (1 - s)f(k^*) = f(k^*) - k^*(n + g + \delta) \tag{22}$$

Solving for the golden rate of consumption entails differentiating (22) and setting this equal to zero:

$$f^{'}(k^*) = n + g + \delta \tag{23}$$

The left hand side of (23) represents the marginal product of capital. The right hand side is the cost of maintaining an additional unit. maximizing consumption requires setting these terms equal.

Because the model includes no utility functions, it is hard to talk about optimal. But it is hard to think of anything except consumption that can effectively stand in for welfare. So we can thus argue that the golden rate of savings is best.

Solving the Cobb-Douglas Version

It is easy to obtain the intensive form of the production function:

$$f(k) = \frac{K^{\alpha}(AL)^{1-\alpha}}{AL} = K^{\alpha}(AL)^{-\alpha} = k^{\alpha} \tag{24}$$

We can then solve for k^* using $(n+g+\delta)k = sk^{\alpha}$:

$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \tag{25}$$

Examination of (25) shows mostly predictable results. Capital (and hence output) per effective unit of labor is increasing in the savings rate and decreasing in depreciation and population growth. It may seem surprising that it is also decreasing in the growth rate of TFP. But capital and output per-worker can be shown to be increasing in g.

$$\hat{k}^* = A(t)k^* = A(0)e^{gt}\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \tag{26}$$

Differentiating with respect to g:

$$\frac{\partial \hat{k}}{\partial g} = A(0)e^{gt}\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}\left(\frac{1}{1-\alpha} + \frac{sg}{n+g+\delta}\right) > 0 \tag{27}$$

Finally, we can solve for s^{gr} by recalling that such a rate yields $f^{'}(k^*) = n+g+\delta$. For the Cobb-Douglas production function, note that $f^{'}(k) = \alpha k^{\alpha-1}$. Inserting (25) and re-arranging yields:

$$s^{gr} = \alpha \tag{28}$$

So the Solow Model makes a clear prediction. A savings rate near $\frac{1}{3}$ maximizes consumption.

Empirical Fit

U.S. per capita GDP is about 50 time sthat of some of the poorer countries in the world. We can check the empirical fit of the Solow Model by seeing if its production function can explain such differences based only on capital differentials. We use use a calibrated Cobb-Douglas production function where $\alpha = \frac{1}{3}$. We first transform the ratio of output into a ratio of capital stocks:

$$\frac{y^r}{y^p} = (\frac{Ak^r}{Ak^p})^{\frac{1}{3}} \tag{29}$$

$$\frac{k^r}{k^p} = (\frac{y^r}{y^p})^3 \tag{30}$$

It thus follows that an output ratio of 50 must correspond to a capital ratio of $50^3 = 125,000$ which is empirically implausible. So the Solow Model alone cannot explain GDP differentials through capital differences alone.

We try to fix this problem by allowing for cross-country differences in TFP. Simply assuming that TFP is exogenously different across countries isn't very interesting. One approach is to endogenize TFP by assuming that firms make a choice about how much to invest in R&D which then yields advances in TFP. This is known as the Endogenous Growth literature. Another approach is to include institutions that allow for TFP differences.

The Solow Model is a good way to start the examination of growth. In addition to having issues with fitting the data, however, it also does not adequately microfound its assumption. To fix this, we can model households' savings-consumption decisions as a result of a formal utility maximization problem. We will do this next. The result is the Infinite Horizon Model.