The Overlapping Generations Model

This model replaces the assumption that households live forever by instead assuming that they live for two periods (generations). The model uses discrete time. We begin with some of the core assumptions:

1. Agents live for two periods. In the first period, they work and save. In the second period, they live off their savings.

2. Agents supply one unit of labor when young.

3. Agents work hard, play hard.

4. Each person gives birth to $(1+n)$ individuals. n is exogenous.

5. The instantaneous utility function is:

$$U(C_i) = \frac{C_i^{1-\theta}}{1-\theta} \tag{1}$$

6. Agents discount the future using $\beta = \frac{1}{1+\rho}$. β is the discount factor while ρ is the discount rate.

7. Production follows $Y_t(K_t, A_tN_t)$ where N_t is the workforce. L_t is the population. Because older people do not work, they are not the same. It follows from #2 and #4 that:

$$N_t = (1+n)N_{t-1} \tag{2}$$

and

$$L_t = N_t + N_{t-1} \tag{3}$$

8. TFP grows at the exogenous rate g so that $A_t = (1+g)A_{t-1}$.

9. The rental rate of capital follows $r_t = f^{'}(k)$. There is no depreciation.

10. The wage rate follows $w_t = f(k) - kf^{'}(k)$.

11. The intensive production function, $f(k)$, has the usual properties including concavity and the Inada Conditions.

Optimization

The representative household maximizes discounted utility over its lifetime:

$$Max_{C_{1t},C_{2t+1}} \quad \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{\beta C_{2t+1}^{1-\theta}}{1-\theta} \tag{4}$$

where C_{1t} indicates the consumption of agents in the first period of life in period t, and C_{2t+1} is the consumption of agents in the second period of life in period $t+1$.

Optimization is straightforward. Consumption in the second period of life is simply savings from the first period:

$$C_{2t+1} = (1 + r_{t+1})(w_t A_t - C_{1t}) \tag{5}$$

where wtA_t is wage income from period t. Note that agents are born owning nothing. They thus receive no income from capital in period t. Only the old own capital in this model.

We can solve this optimization problem in three ways: dynamic programming, a Lagrangian, or using intuition. For shits and giggles, lets go with the third. Consider the following thought experiment:

1. Suppose that in period t, the household increases its consumption by one, infinitesimally small, unit. This increases utility in period t by the marginal utility of consumption which is $C_{1t}^{-\theta}$.

2. As a result of this change, the household saves one unit less. This results in a reduction of $(1 + r_{t+1})$ units of consumption in period $t+1$.

3. The loss in terms of $t+1$ utility to the household is thus $(1 + r_{t+1})C_{2t+1}^{-\theta}$.

4. In time t, however, the household discounts this future utility loss by $\frac{1}{1+\rho}$.
5. It must be true that the costs and benefits of such a change are equal. Otherwise, the household cannot be optimizing. Thus:

$$C_{1t}^{-\theta} = (1 + r_{t+1})C_{2t+1}^{-\theta}\frac{1}{1+\rho} \tag{6}$$

or

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1 + r_{t+1}}{1+\rho}\right)^{1/\theta} \tag{7}$$

Equation (7) yields a pair of interesting results:

1. Whether or not consumption is increasing or decreasing over time depends on the ratio of the market interest rate, r_{t+1} to the subjective discount rate (ρ). If $\frac{r_{t+1}}{\rho}$ is greater than one, then the market is less patient than households. Households thus respond by forgoing some consumption and consumption thus rises over time.

2. The parameter, θ^{-1} is known as the intertemporal elasticity of substitution. It reflects the willingness of household's to transfer consumption from one period to another. Suppose, for example, that $\theta \to \infty$. In this case, $\frac{C_{2t+1}}{C_{1t}}$=1. In words, because intertemporal elasticity equals zero, the household is unwilling to change its consumption at all in response to changes in r_{t+1}. As $\theta \to 0$, however, then even very small changes to the interest rate produce large consumption swings.

We now do some clever tricks to more easily solve the model. Combining (5), the budget constraint, with (6) yields:

$$C_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta}} C_{1t} = A_t w_t \tag{8}$$

Simple re-arranging yields:

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \tag{9}$$

This is just another way of re-writing the household's optimal choice of consumption. It should not be obvious to you that this is a more convenient representation.

You may notice that we are only solving for one of the two representative household's choice of consumption. This is because the choice of old households is trivial, they always consume all of their wealth before they die. We are thus assuming that households don't give a rat's ass about their descendants.

Now recall that $s = (1 - \frac{C_{1t}}{A_t w_t})$, where s is the fraction of income saved Using this and (9) yields:

$$s(r) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta}} (1+r_{t+1})^{\frac{1-\theta}{\theta}} \tag{10}$$

A Quick Aside: Infinite Horizons vs. OLG

The OLG model assumes that households eventually die whereas the Infinite Horizon Model appears to assume that they live forever. At first glance, it seems that the former is more plausible based on the obvious observation that everyone dies. Suppose, however, that the representative household has the following lifetime utility function:

$$U_1t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{\beta C_{2t+1}^{1-\theta}}{1-\theta} + \gamma U_{1t+1} \tag{11}$$

here γ represents the degree of altruism that the generation born in period t has toward the generation born in period $t+1$. The fact that I have delivered more than one lecture at Bates covered in baby puke is empirical evidence that $\gamma > 0$. If $\gamma = \beta$, then households are perfectly altruistic toward their offspring, meaning they care just as much about their kids' utility as their own.

Now iterate (11) forward:

$$U_{2t} = \frac{C_{1t+1}^{1-\theta}}{1-\theta} + \frac{\beta C_{2t+2}^{1-\theta}}{1-\theta} + \gamma U_{1t+2} \tag{12}$$

Equation (12) simply states that the current generation's children will be altruistic toward the current generation's grandchildren. Insert (12) into (11):

$$U_{1t} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{\beta C_{2t+1}^{1-\theta}}{1-\theta} + \gamma U_{1t+1} + \gamma^2 U_{1t+2} \tag{13}$$

The logic of (13) is straightforward. If I care about my child, and if I know that my child cares about their child, then I also care about my grandchildren. We can continue this forward:

$$U_{1t} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{\beta C_{2t+1}^{1-\theta}}{1-\theta} + \gamma U_{1t+1} + \gamma^2 U_{1t+2} + \gamma^3 U_{1t+3} + \gamma^4 U_{1t+4} ... \tag{14}$$

Equation (14) shows that the current generation, because it cares about its kids, behaves as if it lives forever.[1] This is the best interpretation of an infinite horizon model, we are not literally assuming that households live forever, but we are instead assuming that they care about their kids; not an implausible assumption.

[1] Does this explain why I have not slept more than 4 consecutive hours since late October?

Equilibrium Dynamics

As always, our goal is to write the model's equilibrium behavior as something we know how to deal with, in this case a single difference equation. To do this, first note that because the only way to save is to acquire capital, the capital stock in period $t+1$ is just the sum of saving (which is investment) from the previous period:

$$K_{t=1} = s(r_{t+1})L_t A_t w_t \tag{15}$$

We now divide both sides of (15) by $A_t L_t$:

$$k_{t+1} = \frac{s(r_{t+1})w_t}{(1+n)(1+g)} \tag{16}$$

Finally, insert #9 and #11 to eliminate r_{t+1} and w_t:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f^{'}(k_t))[f(k) - kf(k)] \tag{17}$$

Equation (17) is as far as we can go with general utility and production functions. A closed form solution (with just exogenous terms on the right hand side) does not generally exist. k_{t+1} is said to be determined by (17) implicitly because it also appears on the right hand side. We now, consider, five potential dynamics depending on the type of utility and production functions.

#1: A unique stable interior equilibrium.

One example of such an equilibrium is logarithmic utility ($\theta = 1$), and Cobb-Douglas Production ($y = k^\alpha$). Note that (10) reduces to $s(r) = \frac{1}{2+\rho}$. Likewise $f(k) - kf(k) = (1-\alpha)k_t^\alpha$. Combining these and (17) yields:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)k_t^\alpha \tag{18}$$

The dynamics of (18) may be illustrated by a phase diagram with k_{t+1} on the vertical axis and k_t on the horizontal. Where the function intersects a 45 degree line, a steady state exists.

Graph: Phase Diagram #1

The dynamics of the model in this case are similar to the Infinite Horizon Model. A unique interior steady state exists, and this steady state is stable from either direction.

There is, however, one important difference between this equilibrium and that of the Infinite Horizon Model. Recall that the latter assumed complete an competitive markets so that the equilibrium path was always efficient. This is not generally the case in the OLG model. Consider this case with the further simplifying assumption that $g = 0$. Using (18), the steady state may be written as:

$$k^* = \left[\frac{1-\alpha}{(1+n)(2+\rho)} \right]^{\frac{1}{1-\alpha}} \tag{19}$$

It then follows that the marginal product of capital at the steady state is:

$$f^{'}(k^*) = \alpha(k^*)^{\alpha-1} = \frac{\alpha(1+n)(1+\rho)}{1-\alpha} \tag{20}$$

In the Solow Model, the condition for maximizing steady state consumption is $f^{'}(k^*) = n$. This condition is unchanged for the OLG model. Note that (20) may thus imply that the marginal product of capital is either too high or too low.

In general, discussing welfare in the OLG model requires that we place weights on the utilities of different generations. This analysis skirts this issue, however, by examining the steady state where all generations have the same consumption.

The specific incomplete market is abstract. The problem is that there is no market for one generation to contract with later generations. Each generation chooses its consumption (and thus the next period's capital stock) without factoring in the effect of changing the capital stock on the next generation. This is a distortion. Note, however, that it no longer exists if households are perfectly altruistic so that $\beta = \gamma$.

#2: Multiple Stable Steady States.

It is also possible that the model yields the following phase diagram.

Graph: Phase Diagram #2

In this example there are four steady states, including zero. Two of the interior steady states are stable. This example offers a fundamentally different explanation for the dramatic cross country income differentials observed in the data. Rather than just being at different points on a common path to a common steady state developed economies may be in the neighborhood of a higher steady state than poor countries.

Multiple steady states is one type of *multiple equilibria*, one of the hottest current research topics in macroeconomics.

There are several sets of assumptions that yield this case. Suppose, for example, that for low enough levels of k, the production function has the usual form. For high values of K, however, the model begins to exhibit increasing returns due to production spillovers.

#3: The Zero Outcome.

Graph: Phase Diagram #3

Here, the capital stock (and hence output and consumption) converges toward zero. This case requires that as capital goes to zero, the savings function is flatter that the 45 degree line. This may happen if the marginal product of capital does not satisfy the Inada conditions and is thus sufficiently flat for very low levels of capital.

#4: Boom or Bust.

Graph: Phase Diagram #4

In this case, the unique interior steady state is unstable. For low values of capital, the model converges toward zero. Above this steady state, however, k diverges toward infinity.

#5: Sunspots.

Graph: Phase Diagram #5

This phase diagram is similar to #2 except that the relationship (which is not a function in this case) from (17) is backward bending for part of the space. Here multiple values of k are consistent with optimization. This is a second type of multiple equilibria where the path around a steady state is indeterminate.

This case allows for the possibility of *sunspots*. Suppose that two different values of k are consistent with optimization. In this case, it is fully possible that agents will base their choice of path on extraneous economic variables or sunspots.

Suppose that both k_l and k_h are equilibrium paths. The household may rationally choose the former if the Chairman of the Federal Reserve took his coffee black, while other wise choosing the latter. This is not generally an efficient phenomenon.

Sunspots are an especially important part of business cycle theory. We will discuss them in more depth during that part of the course.