The Infinite Horizon Model[1]

This model adds microfoundations to a growth model. Although there are other differences with the Solow Model, the main one is that the savings-consumption choice results from rational households maximizing their utility rather than exogenously saving a fraction of their income.

There are two fundamental ways in which we can solve this type of model. First, we can imagine a *social planner* who chooses aggregate variables such as consumption, labor supply, etc. Second, we can solve for the *decentralized equilibrium* which consists of individual agents and firms maximizing subject to constraints which include prices. The former approach, by definition, yields a Pareto optimal allocation. The second approach yields the same result as the first if and only if there are complete and competitive markets (no market power, externalities, public goods, etc.) In general, the two thus yield different results.

For this model, markets are complete and competitive and either approach may thus be used.

Representative Agents

This model relies on *representative agents*. This approach assumes that all households, for example, are identical. If they are identical, and if a unique solution to their optimization problem exists, then one household's choice will be the same as all others. Suppose for example, that the representative household chooses $c_i(t) = 7$.

For mathematical convenience we want the household to be so small that it has no effect on any price in the economy, Otherwise, we would have to factor these effects into its optimization problem. We thus assume that a continuum of identical households exist on the unit interval. Aggregate consumption is thus the sum of individual consumption:

$$c(t) = \int_0^1 c_i(t) di \tag{1}$$

If all agents are identical, then integration yields $c_i(t) = c(t)$. Thus solving for the representative agent's choices is the same as solving for the aggregate values. We thus typically dispense with the i subscripts.

[1]These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

There are two potential drawbacks with the representative agent approach. First, it can be confusing to think of agents as price takers when they seem to be the only such agent in the model. It is thus important to remember that the representative agent is just one of a very large number of similar agents. Second, representative agents can make it difficult to think about heterogeneity.

Firms

A continuum of identical profit maximizing firms exists. The production function is: $Y = F(K, AL)$, with the same assumptions as the Solow Model. As with the Solow Model, TFP grows at the exogenous rate g. For simplicity, we set $\delta = 0$.

A profit maximizing firm chooses inputs so that their prices equal their marginal products. For capital, they thus set the following:

$$r(t) = F_K = \frac{\partial F(K(t), A(t)L(t))}{\partial K(t)} \tag{2}$$

Recall from Chapter 1 that $F(K, AL) = ALf(\frac{K}{AL})$. Differentiating both sides by K yields, $\frac{\partial F(K,AL)}{\partial K} = ALf^{'}(\frac{K}{AL})(\frac{1}{AL}) = f^{'}(k)$. We may thus re-write (2) so that:

$$r(t) = f^{'}(k) \tag{3}$$

The representative firm's labor supply choice is similar:

$$W(t) = F_L \tag{4}$$

It will be helpful, however, to rewrite the wage in terms of capital so that we can represent the model with fewer equations. Note that the total amount of income paid to capital is $r(t)K(t) = f^{'}(k(t))K(t)$. Likewise, the total amount of income paid to labor is $W(t)L(t) = F_l L(t)$. These must add up to total income in the economy:

$$F(K, AL) = f^{'}(k(t))K(t) + F_L L(t) \tag{5}$$

Dividing both sides of (5) by AL yields:

$$f(k) = \frac{F_L}{A} + f^{'}(k(t))k(t) \tag{6}$$

Finally, using (4) and re-arranging yields:

$$W(t) = A(t)\left[f(k(t)) - k(t)f^{'}(k(t))\right] \tag{7}$$

Later, it will also be helpful to work with the wage per unit of effective labor:

$$w(t) = \frac{W(t)}{A(t)} = \left[f(k(t)) - k(t)f^{'}(k(t))\right] \tag{8}$$

Households

The representative household's problem is complex. Rather than maximizing utility at any one time, it maximizes its lifetime utility which is the discounted stream of utility today and infinitely far into the future:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt \tag{9}$$

Note that this model assumes that the representative household (hereafter "RA") lives forever. This assumption has important implications. We will discuss these, and the plausibility of this assumption, later on.[2] The parameter ρ is the discount rate. As it increases, households become less patient and care less about future utility relative to today.

The function $u(C(t))$ is the *instantaneous utility function*. It indicates the utility that the RA gets at any given time. Again, keep in mind that the RA is not maximizing this. $C(t)$ is the consumption for each member of the household. $L(t)$ is the population (we are assuming that the population equals the labor force), and H is the number of members for each household. Total instantaneous utility thus equals $u(C(t))\frac{L(t)}{H}$.

We further assume that:

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \tag{10}$$

where $\theta > 0$ and $\rho - n - (1-\theta)g > 0$. This is known as both a *constant-relative-risk aversion* and *constant elasticity of substitution* utility function. Note that if $\theta = 1$, then $u(C(t)) = ln(C(t))$.

The RA faces an intertemporal budget constraint that requires that its discounted stream of expenditures equal its initial assets plus its discounted stream of income. Define $R(t) = \int_{\tau=0}^{t} r(\tau)d\tau$, as the cumulative discounting that occurs between times 0 and t:

[2] It is not as unrealistic as it seems at first glance. We will see that a model where agents live for a finite amount of time and care as much about their children as themselves, yields the same results.

$$\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} dt \tag{11}$$

There is another notable feature of (11), there are no expectations. This is because the model contains no uncertainty, such as random shocks. All agents thus know the future with certainty.

It is challenging to work directly with (11). Fortunately, there is a short-cut. In the limit as $t \rightarrow \infty$, this budget constraint is satisfied as long as the RA maintains a positive discounted level of capital. We thus rely on the following equation instead of (11):

$$lim_{s \rightarrow \infty} e^{-R(s)} K(s) \geq 0 \tag{12}$$

This is known as a *no-Ponzi game condition.* It forbids the RA from rolling over its debt indefinitely.

It will be useful to work with consumption per unit of effective labor. We thus define $c(t) = \frac{C(t)}{A(t)}$. Using this definition, we can re-state the instantaneous utility function:

$$\frac{C(t)^{1-\theta}}{1-\theta} = \frac{[A(t)c(t)]^{1-\theta}}{1-\theta} \tag{13}$$

Recall that, because TFP evolves according to an ordinary differential equation that $A(t) = a(0)e^{gt}$. Inserting this into (13) yields:

$$\frac{C(t)^{1-\theta}}{1-\theta} = A(0)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \tag{14}$$

Inserting (14) into (9) and noting that $L(t) = L(0)e^{nt}$ yields:

$$U = A(0)^{1-\theta} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-\rho t} e^{(1-\theta)gt} e^{nt} \frac{c(t)^{1-\theta}}{1-\theta} dt \tag{15}$$

Now define $B = A(0)^{1-\theta} \frac{L(0)}{H}$ and $\beta = \rho - n - (1-\theta)g$. β includes household's inherent impatience, captured by the inclusion of ρ, as well as population growth and technological progress:

$$U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \tag{16}$$

The next step is to re-state the budget constraint in terms of units of effective labor. First, re-state (11) as follows:

$$\int_{t=0}^{\infty} e^{-R(t)} \frac{A(t)L(t)}{H}(c(t) - w(t)) \leq k(0)\frac{A(0)L(0)}{H} \tag{17}$$

Note that the left hand side of (17) was converted to units by effective labor by multiplying by $\frac{A(t)}{A(t)}$. The term in the denominator disappears when consumption and the wage rate are re-stated in units per effective unit of labor. The term in the numerator remains. the same trick is used on the right hand side, except that we multiply by $\frac{A(0)}{A(0)}$.

Now, use our primer on difference equations to note that:

$$A(t)L(t) = A(0)L(0)e^{(n+g)t} \tag{18}$$

Inserting (18) into (17) yields the most useful representation of the budget constraint:

$$\int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t}(c(t) - w(t)) \leq k(0) \tag{19}$$

We can also re-state the no-ponzi games condition, (12):

$$lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) \geq 0 \tag{20}$$

Optimization

The first step in solving the RA's optimization problem is to note that, because utility is always increasing in consumption, the budget constraint will always hold with equality.

We then use a *Lagrangian Multiplier* to obtain a first-order condition.

$$\mathcal{L} = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt = \lambda[k(0) - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t}(c(t) - w(t))] \tag{21}$$

This method adds the constraint to the optimization problem. The Lagrangian Multiplier, λ, represents the value to the RA of relaxing the constraint by giving the RA one additional unit of capital. Optimization then consists of differentiating (21) with respect to $c(t)$. Doing so yields:

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)} e^{(n+g)t} \tag{22}$$

The remaining steps are more clever than difficult. First, take logs of (22)

$$ln(B) - \beta t - \theta ln(c(t)) = ln(\lambda) - R(t) + (n+g)t = ln(\lambda) - \int_{\tau=0}^{6} r(\tau)d\tau + (n+g)t \qquad (23)$$

Equation (23) must be true at all times. This implies that the derivatives with respect to time of each side must be equal. We thus differentiate:

$$-\beta - \theta\frac{\dot{c}(t)}{c(t)} - r(t) + (n+g)t \qquad (24)$$

note that we are using the property of logs where $\frac{\partial ln(x)}{\partial x} = \frac{\dot{x}}{x}$.

Finally, we can clean up (24) by eliminating β and $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f^{'}(k(t)) - \rho - \theta g}{\theta} \qquad (25)$$

Equilibrium

To close the model, note that the capital accumulation takes the following form:

$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t) \qquad (26)$$

As with the Solow Model, capital (per effective unit of labor) is lost due to population growth, technical progress, and depreciation, although the latter term is set to zero for simplicity. New capital equals investment which is simply output less consumption.

The system is thus entirely represented as a $2x2$ system of differential equations, (25) and (26), along with the no-ponzi games condition, (20).

We begin by examining the dynamics of c. Suppose that k is very close to zero. From our assumptions about the production function, $f^{'}(k)$ is very large. From (25), consumption is increasing. As k increases, $f^{'}(k)$, decreases, approaching zero as $k \to \infty$. It thus follows that for some amount of capital, k^{*}, consumption is constant ($\dot{c} = 0$). We can plot this relationship on a phase diagram:

Graph: Dynamics of c

We now consider the dynamics of k. Equation (26) implies that capital is increasing if and only if $f(k(t)) - (n+g)k(t) = c(t)$. This is the break even level of consumption. Note the following:

i. If $k = 0$, then the break even level equals zero.

ii. For low levels of k, $f^{'}(k) > n+g$. Here, increasing the capital stock thus increases the break even level of consumption.

iii. Once $f^{'}(k) = n+g$, the break even level of consumption is maximized.

iv. Above this level, increasing the capital stock reduces the break even level of consumption.

Graph: Dynamics of k

The full dynamics of the model may be analyzed by examining a phase diagram which combines the dynamics of both consumption and capital.

Graph: Phase Diagram

The Saddle Path

Suppose that the initial level of capital is less than the steady state value, $k(0) < k^*$. Using the phase diagram we can consider how the economy will adjust for different initial choices of c.

Graph: Phase Diagram

A. Here, initial consumption is very high. Consumption increases which causes further depletion of the capital stock.

B. Here, the capital stock is initially stable. Increasing consumption, however, eventually causes a path similar to A.

C. Here, both consumption and capital are initially increasing. Eventually, however, the capital stock begins to decline

D. Here, consumption is very low which causes the capital stock to grow to a maximum value where no consumption occurs.

F. Here, consumption and capital increase so that the economy converges toward the steady state. This path, known as a *saddle path* is unique. If consumption is any higher, then the economy will follow a path like C. If it is any lower, then it will follow a path like D.

Here is where the no-ponzi game condition becomes important. Clearly, any initial choice of c that yields a path like D cannot be optimal because the household could clearly increase its consumption in every period by choosing a slightly higher initial value of consumption. It is also the case that any initial choice of consumption above F violates the no-ponzi game condition because capital goes to negative infinity. It thus follows that optimality requires that the RA always choose the initial value of consumption so that the economy lies on the saddle path.

A Parameter Change

Suppose that either ρ or θ increase. From (25), this change will reduce the value of k^*, the steady state capital stock. Supposing that the economy is initially at the steady state, the previous steady state level of consumption will initially be well above the saddle path.

The previous analysis shows that optimality requires that the RA immediately increases its consumption so that the economy is on the new saddle path. Consumption and capital will then decline over time as the economy converges toward the new steady state.

Collectively, it is useful to compare these results to those of the Solow Model:

1. The most important theoretical distinction between the Solow Model and the Infinite Horizon Model is that households are optimizing in the latter but not generally in the former. Hence, there is no golden rate of savings in this model.

2. The concept of optimality itself was ill-defined in the Solow Model. The golden rate of saving yielded the highest level of steady state consumption. But we had nothing to say about optimal adjustment if the economy started away from the steady state in the Solow Model. Here, households choose the optimal path.

3. Because there are no distortions from complete and competitive markets, the households choices are Pareto efficient in the Infinite Horizon Model.

Algebra

In solving the infinite Horizon Model, we have relied on the phase diagram. Alternatively, we could have relied on algebra. Suppose, for example, that the model could be represented as a linear system of 2 difference equations:

$$\begin{bmatrix} \dot{c} \\ \dot{k} \end{bmatrix} + A \begin{bmatrix} c(t) \\ k(t) \end{bmatrix} = 0 \tag{27}$$

This model is non-linear. But it is possible to take a linear approximation of this model and thus obtain A from (27). If so, we would then find that there is one eigenvalue of A less than zero and one greater than zero. this suggests a dynamically unstable system. Suppose, however, that we took the eigendecomposition of A, where $A = S\Lambda S^{-1}$. Pre-multiplying both sides of (27) by S^{-1} then yields:

$$\dot{z} + \Lambda z(t) = 0 \tag{28}$$

Now consider the root that is less than one and that is therefore resulting in instability. Suppose, for example, that it is λ_1. It then follows that:

$$\dot{z_1} = -\lambda_1 z(t)_1 \tag{29}$$

But as we have already seen, any choice of $c(t)$ that results in explosive behavior is sub-optimal. Suppose, however, that the RA chooses $c(t)$ so that $z(t)_1 = 0$. In this case, the system will not be explosive.

In this example, $z(t)_1$ is the saddle path. Because it is a linear combination of the two variables, it may be written as $c(t) = \chi k(t)$. It is the unique choice of consumption that allows the economy to converge back toward its steady state.