ECO 318: Dynamic Programming[1]

Dynanic programming is a very powerful mathematical tool that can be used to solve many macroeconomic models. We will begin with some general terminology and then consider some macroeconomic examples.

Consider the following problem. A household must choose a sequence of a *control variable*, u_t. A good example is consumption, which we think of as a choice that households make. The household wants to maximize some objective function, an example would be a utility function:

$$\sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{1}$$

The variable x_t is known as a *state variable*. The household does not choose the state variable. Instead it takes its value in the current period as given. It evolves, however, according to some function that depends on the lagged state and control variable:

$$x_{t+1} = g(x_t, u_t) \tag{2}$$

As an example, capital is typically a state variable, households do not choose its current value. Its future value, however, depends on consumption (a control) from the prior period, as well as lagged capital. Dynamic programming attempts to find a timeless solution to this problem. It may be represented as a *policy function*:

$$u_t = h(x_t) \tag{3}$$

The policy function maps from the state variable(s) to the optimal value of the control(s). If such a function exists, then if I know the state variables (and exogenous shocks), then I know the optimized controls. If such a solution exists, the problem is said to be *recursive*.

Suppose that the household is making its choice in period 0. Denote x as the current period value of the state and $\tilde{x}$ as next period's. The key is to define the *value function*, $V(x_0)$, the

[1] These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

optimized value from (1). As an example, the value function may be the level of a household's discounted stream of utility over the infinite horizon. We can write the value function as:

$$V(x_0) = max_{u_s} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{4}$$

Suppose we knew $V(x_0)$ (which, at this time, we do not). We could then solve for the policy function by differentiating the following with respect to the control variable:

$$max_u \{r(x, u) + \beta V(\tilde{x})\} \tag{5}$$

Finally, we do some substituting to re-write what we now call the *Bellman Equation*.

$$V(x) = max_u \{r(x, u) + \beta V[g(x, u)]\} \tag{6}$$

we can then insert the policy function into (6):

$$V(x) = \{r(x, h(x)) + \beta V[g(x, h(x)]\} \tag{7}$$

where we have dropped the max because it is implied by the policy function. It is normal to be lost here, we will need examples for this to start making sense. But the usefulness of (7) is that we have eliminated the control from the problem. We just need to solve for the value function only as a function of the state variable.

There are numerous ways to solve for the value function. For large models, we would typically use computational methods. For simple models, guess and verify will be our method of choice. We illustrate this with an example.

Optimal Growth

Consider a household that wishes to maximize its lifetime (infinite) utility. Preferences are logarithmic and the discount factor is β :

$$\sum_{t=0}^{\infty} \beta^t ln(c_t) \tag{8}$$

Consumption is the control variable. Capital is the state variable. It evolves according to:

$$k_{t+1} = Ak_t^{\alpha} - c_t \tag{9}$$

This capital accumulation equation, not unlike what you saw in ECO 270, is an example of the g function in the general notation. Our goal is to solve fir the value function as a function of capital, $V(k)$.

Our method is guess and verify. We will guess that the value function has a specific functional form, and then we will try to solve for the coefficients. It may take more than one try.

For now, we will guess correctly. We guess the following functional form:

$$V(k) = E + Fln(k) \tag{10}$$

The trick is to write the Bellman Equation as a function of capital only. We will thus use (9) to eliminate consumption. The first art of the right hand side of (7) is the current level of utility:

$$V(k) = ln(Ak^{\alpha} - \tilde{k}) + \beta V[\tilde{k})] \tag{11}$$

The second term on the right hand side is our guessed value function, but for the next period:

$$V(k) = ln(Ak^{\alpha} - \tilde{k}) + \beta E + \beta Fln(\tilde{k}) \tag{12}$$

A tricky point is that we can think of the control variable either as c_t or k_{t+1}. This is valud because by choosing consumption today, we pin down the capital stock tomorrow. We can thus get a policy function by differentiating (12) with respect to k_{t+1} and setting this term equal to zero.

$$\frac{-1}{Ak^{\alpha} - \tilde{k}} + \frac{\beta F}{\tilde{k}} = 0 \tag{13}$$

re-arranging:

$$\tilde{k} = \frac{\beta F}{1 + \beta F} Ak^{\alpha} \tag{14}$$

Some grisly algebra finishes the problem off. We use (14) to get our Bellman Equation. Note that:

$$c = Ak^{\alpha} - \tilde{k} = Ak^{\alpha} - \frac{\beta F}{1+\beta F}Ak^{\alpha} = \frac{1}{1+\beta F}Ak^{\alpha} \tag{15}$$

We can then re-write the Bellman equation as current utility plus next period's value function:

$$V(k) = ln(\frac{1}{1+\beta F}Ak^{\alpha}) + \beta E + \beta F ln(\frac{\beta F}{1+\beta F}Ak^{\alpha}) \tag{16}$$

We now need to verify that the functional form matches our guess and we are done. Expanding (16) yields:

$$V(k) = ln(\frac{A}{1+\beta F}) + \beta F ln(\frac{A\beta F}{1+\beta F}) + \alpha ln(k) + \beta F \alpha ln(k) \tag{17}$$

So the right hand side of (17) matches our guess, the value function includes only a constant and logged capital. We can solve for F:

$$F = \alpha + \beta F \alpha = \frac{\alpha}{1-\beta\alpha} \tag{18}$$

We could also solve For E as some god-awful function of the model's parameters but it is not worth the effort. We conclude by combining (15) and (18) to get consumption as a function of capital:

$$c = \frac{1}{1+\beta F}Ak^{\alpha} = (1-\beta\alpha)Ak^{\alpha} \tag{19}$$

or, using more familiar notation:

$$c_t = (1-\beta\alpha)Ak_t^{\alpha} \tag{20}$$

And we are done, if we know the state, then we know the optimal value of the control. This is hard. I doubt you will fully get it at this point. A homework assignment will take you through another example.

Guessing Wrong

It isn't uncommon to have to guess more than one time. Suppose that we made the following guess instead:

$$V(k) = E \tag{21}$$

We have erred by omitting logged-capital from our value function. Our Bellman Equation is now:

$$V(k) = ln(Ak^{\alpha} - \tilde{k}) + \beta E \tag{22}$$

Here, we want to maximize (22) through our choice of $\tilde{k}$. But we have a corner solution where $\tilde{k} = 0$. In other words, if next period's value function does not depend on anything I do, I will consumer everything today so that no capital is left next period. It is thus the case that $c = Ak^{\alpha}$. Inserting this into the Bellman Equation:

$$V(k) = ln(Ak^{\alpha}) + \beta E = ln(A) + \beta E + \alpha ln(k) \tag{23}$$

But verification fails. The value function from (23) includes logged capital and does not match our guess. We need to try again, Usually, we would add the missing term to our next guess. Here, doing so would give us the correct solution on our second try.

The power of the value function is that, if we can solve for it, we then know the model's control variables as a function of the model's state variables and shocks. This then makes it easy to simulate. Solving value functions, however, is not easy, especially as model's become large.

Euler Equations