ECO 318: A Primer on Difference and Differential Equations[1]

Almost all macroeconomic models can be represented as systems of differential equations (if time is continuous) or difference equations(if time is discrete: 2008, 2009, 2010, etc.). We thus proceed with a primer on this material. It is quite challenging. So I strongly encourage you to struggle with this material as soon as possible. If you can acquire a passable working knowledge of these techniques, then you should be in good shape as we begin the heart of the class.

Continuous vs. Discrete Time

Most modern macroeconomic models are *dynamic* in that current endogenous variables depend on past, or expected future values. Time is thus an important component of these models. There are two basic ways to model time. The first is *continuous time*, where t may take any value. The second is *discrete time* where t is defined over fixed period lengths such as years, quarters, days, etc.

I have long thought of continuous time as more realistic and discrete time as a useful approximation. When I mentioned this to a colleague whom I respect, however, he was surprised and told me that he holds the exact opposite view.

Univariate Difference Equations

Difference equations apply to discrete time. A difference equation is simply an equation where the current value of a variable depends on either the future or past. A *first order difference equation* contains only 2 periods. We will focus on these and rely on an example. Suppose that:

$$y_t = a + by_{t-1} \tag{1}$$

further suppose that we know y_0, the value of y in some initial period.

[1]These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

Solving this difference equation may be done through backwards iteration. The first step is to note that if (1) is true, then by back-dating this equation, it is also true that:

$$y_{t-1} = a + by_{t-2} \tag{2}$$

It is then possible to insert (2) into (1):

$$y_t = a + b(a + by_{t-2}) = a + ab + b^2 y_{t-2} \tag{3}$$

Backwards iteration thus moves back the period that appears on the right hand side by one period. We can do this again by noting that:

$$y_{t-2} = a + by_{t-3} \tag{4}$$

Inserting (4) into (3) yields:

$$y_t = a + b(a + by_{t-2}) = a + ab + b^2(a + by_{t-3}) = a + ab + ab^2 + b^3 y_{t-3} \tag{5}$$

We continue iterating backwards t periods. Seeing a pattern, this yields:

$$y_t = a(1 + b + b^2 + b^3 b^t) + b^t y_0 \tag{6}$$

There is no fixed number of iterations that we must do to see the pattern. Some people are better at seeing the pattern than others.

Equation (6) thus represents the solution to the difference equation. Note that we need the initial condition, y_0, to obtain this solution.

In this example, b is the root of the equation. There are two important characteristics of b:

i. If $|b| < 1$, then the difference equation is *stationary*.[2] Stationarity implies that the system is

[2] Stationarity is actually an important and more complicated concept, especially for time-series econometrics. Take this just as a temporary working definition.

mean-reverting. To see this, suppose that $a = 0$, $b = \frac{1}{2}$, and $y_0 = 1$. It then follows that $y_1 = \frac{1}{2}$, $y_2 = \frac{1}{4}$, $y_3 = \frac{1}{8}$... y_t clearly converges back to its mean of zero.

If $|b| > 1$, however, then the system is *explosive* implying that it diverges to either infinity or negative infinity. To see this, suppose that $a = 0$, $b = 2$, and $y_0 = 1$. It then follows that $y_1 = 2$, $y_2 = 4$, $y_3 = 8$...

Finally, if $|b| = 1$, then it is neither stationary nor explosive. This is known as a *random walk*.

ii. If $b < 0$, then the equation *oscillates* meaning that it alternates being above and below its mean. To see this, suppose that $a = 0$, $b = -\frac{1}{2}$, and $y_0 = 1$. It then follows that $y_1 = -\frac{1}{2}$, $y_2 = \frac{1}{4}$, $y_3 = -\frac{1}{8}$... y_t converges back to its mean of zero, but it alternates between being above and below it.

If $b > 0$, then then y_t does not oscillate.

Systems of Difference Equations

Suppose that we have the following system of difference equations:

$$\pi_t + \alpha y_t = \beta \pi_{t-1} + \gamma y_{t-1} \tag{7}$$

$$\pi_t + \omega y_t = \chi \pi_{t-1} + \zeta y_{t-1} \tag{8}$$

Which, based on the previous topic, may be written as a single matrix equation:

$$X_t = C X_{t-1} \tag{9}$$

where C is a $2x2$ matrix. It may be a good idea for you to solve for C if you are still unsure of some of the material from the linear algebra primer.

Recall that C may be written using the eigendecomposition:

$$C = S\Lambda S^{-1} \tag{10}$$

where λ is the matrix (on the diagonal elements) of eigenvalues and S is the matrix of corresponding eigenvectors.

Define $Z_t = S^{-1}X_t$.

Note that S is $2x2$ and X_t consists of a vector that includes π_t and y_t. z_t is therefore $2x1$. Each element is a *linear combination* of π_t and y_t. For example, $3\pi_t + 2y_t$ is an example of a linear combination.

The system can then be written as:

$$X_t = S\Lambda S^{-1}X_{t-1} \tag{11}$$

Now pre-multiply both sides by S^{-1}

$$S^{-1}X_t = S^{-1}S\Lambda S^{-1}X_{t-1} \tag{12}$$

Or:

$$Z_t = \Lambda Z_{t-1} \tag{13}$$

Recall that:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{14}$$

Matrix multiplication then yields two equations:

$$Z_{1,t} = \lambda_1 z_{1,t-1} \tag{15}$$

$$Z_{2,t} = \lambda_2 z_{2,t-1} \tag{16}$$

So the 2x2 system may be written as a pair of independent difference equations. The two roots are thus λ_1 and λ_2, the two eigenvalues of C. Independence is useful here because it allows us to examine the dynamic properties of each equation in isolation.

The eigenvalues thus govern the stability of the system. The system is stationary if and only if both eigenvalues have absolute value less than one. If either eigenvalue is greater than one in absolute value, then the system is explosive.

Differential Equations

Suppose that we have the following equation:

$$\dot{y} = \frac{\partial y}{\partial t} + ay = 0 \tag{17}$$

It can be shown[3] that the solution to this equation is:

$$Y(t) = y(0)e^{-at} \tag{18}$$

where $e \approx 2.718$, the exponential function.

We can perform the same type of stability analysis for differential equations:

i. Suppose that $a > 0$. In this case, as $t \to \infty$, $e^{-at} \to 0$. The system thus converges back to zero and is thus stationary.

Suppose, however, that $a < 0$. In this case, as $t \to \infty$, $e^{-at} \to \infty$. The system is thus explosive.

Systems of difference equations may be handled in much the same way as systems of difference equations, Now, however, all of the eigenvalues must be positive (if the matrix system follows (17)) for the system to be stationary.

Forward Looking Difference Equations

[3] take a course in differential equations for the exciting details

Any difference equation may be written as either a forward or a backward looking difference equation . Recall (1):

$$y_t = a + by_{t-1} \tag{19}$$

Re-arranging this equation yields:

$$y_{t-1} = -ab^{-1} + b^{-1}y_t \tag{20}$$

Re-dating by moving all time subscripts and re-labeling then yields:

$$Y_t = \alpha + \beta y_{t+1} \tag{21}$$

where $\alpha = -ab^{-1}$, and $\beta = b^{-1}$. Note that the absolute value of b is less than one if and only if the absolute value of β is greater than one. Thus, for forward looking difference equations, stationarity requires that $|\beta| > 1$. This confuses almost everyone studying this material for the first time.

An Example

Suppose that we are given the following pair of difference equations:

$$x_t = x_{t-1} + \frac{1}{2}y_{t-1} \tag{22}$$

$$y_t + \beta x_t = y_{t+1} \tag{23}$$

and that we wish to understand how the stationarity of the system depends on β. The first complication is that one of the equations is backward looking while the other is forward looking. We can thus start by re-writing one of the equations. I will choose the latter although it can work either way:

$$y_t = y_{t-1} + \beta x_{t-1} \tag{24}$$

This is a reminder that we can always re-date a difference equation as needed. The next step is to write the system in matrix form:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} \tag{25}$$

Recall that the roots of the system are just its eigenvalues. So we must take the determinate of the following matrix and set it equal to zero:

$$\begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \beta & 1-\lambda \end{bmatrix} \tag{26}$$

This is solved by:

$$(1-\lambda)^2 - \frac{1}{2}\beta = 0 \tag{27}$$

Once again, the roots are solved using the quadratic formula:

$$\lambda_i = \frac{2 \pm \sqrt{2\beta}}{2} \tag{28}$$

I am not too concerned with the final steps. But note that this reduces to 1 plus or minus something. It is thus impossible for both eigenvalues to be less than one in absolute value. The system is always explosive.