Stochastic Processes: Key

Consider the following AR(2) process

$$x_t = \delta + \alpha_2 x_{t-2} + u_t \tag{1}$$

1. We may first re-date (1):

$$x_{t-2} = \delta + \alpha_2 x_{t-4} + u_{t-2} \tag{2}$$

Inserting (2) into (1):

$$x_t = (1+\alpha_2)\delta + \alpha_2^2 x_{t-4} + u_t + \alpha_2 u_{t-2} \tag{3}$$

which is an ARMA(4,2). Iterating again yields:

$$x_t = (1+\alpha_2+\alpha_2^2)\delta + \alpha_2^3 x_{t-6} + u_t + \alpha_2 u_{t-2} + \alpha_2^2 u_{t-4} \tag{4}$$

Note that each iteration yields an additional constant term and moves the lag back two periods. It also adds an additional error term. Thus for n iterations:

$$x_t = (1+\alpha_2+\alpha_2^2 ... + \alpha^n)\delta + \alpha_2^n x_{t-2n} + u_t + \alpha_2 u_{t-2} + \alpha_2^2 u_{t-4} ... + \alpha_2^n u_{t-2n} \tag{5}$$

this may be written as an MA(∞) if and only if $|\alpha_2| < 1$, in which case the autoregressive term disappears.

2-4. The true mean is time independent if and only if $|\alpha_2| < 1$. In this case:

$$E[x_t] = \frac{\delta}{1-\alpha} + \frac{\bar{u}}{1-\alpha} \tag{6}$$

The true mean is also time independent if and only if $|\alpha_2| < 1$. In this case:

$$V[x_t] = \sigma_u^2(1+\alpha_2^2+\alpha_2^4 ...) = \frac{\sigma_u^2}{1-\alpha^2} \tag{7}$$

The process is weakly stationary if and only if $|\alpha_2| < 1$.

Notice that #1-#4 are not substantively different than from an AR(1) process. This is because, by assuming that (1) depends only on the second lag, it is not substantively different than an AR(1). Suppose, for example, that we have a variable measured annually that is AR(1).

If we re-define a period as every six months, then we can think of period t as depending on $t-2$ but not $t-1$ where it is measured.

5. I have an ulterior motive in asking this question. I don't expect you to derive these conditions. The wording of the question, however, (which does not include terms such as "derive" or "show") makes it entirely fair for you to find, and cite, a source that does obtain these conditions. When doing reserach, this is how we usually answer questions, we don't re-derive everything ourselves.

It isn't too hard to derive these conditions, but it does take some linear algebra which we have not yet covered. They are:[1]

$$|\alpha_2| < 1 \tag{8}$$

$$\alpha_1 + \alpha_2 < 1 \tag{9}$$

$$\alpha_2 - \alpha_1 < 1 \tag{10}$$

6. False. All finite MA processes are stationary. But, as shown in class, we have seen that any AR(1) process may be re-written as a MA(∞) process. This includes non-stationary AR(1) processes.

7. When it is stationary, a unique fixed point will be the true mean

$$x = \delta + (\alpha_1 + \alpha_2)x + \bar{u} \tag{11}$$

$$x = \frac{\delta + \bar{u}}{1 - \alpha_1 - \alpha_2} \tag{12}$$

You may also not that here, I am denining the mean of the error term to be $\bar{u}$, while in class we have assumed this is zero. It doesn't really matter. Allowing $\bar{u} \neq 0$ is the same as changing the constant from the case where $\bar{u} = 0$.

8. No. In general, we do not expect a sample mean to equal the true mean. If the process is mean-ergodic (and this one is), then as $T \to \infty$, the sample mean will converge to the true mean. But this will generally not be true for a a finite sample seize, even if it is mean-ergodic.

[1] See, for example, Linton, O. and R. McCrorie. 1995. "Differentiation of an Exponential Matrix Function." *Econometric Theory*, Vol. 11, No. 5, Symposium Issue: Trending Multiple Time Series , pp. 1182-1185.