An Introduction to Time Series[1]

These notes represent an introduction to time series. Much of the material is conceptional and some of it is review. This course's focus is on estimation and it will be heavily example based. Nevertheless, we need this material as a foundation.

Suppose that there exists a data generating process (DGP). For example, the U.S. economy is the data generating process for variables such as GDP, inflation, etc. A *time series* is a realization of the data generating process. We can denote some time series as $\{x_t\}_{t=1}^{T}$. This notation is simply a collection of some variable running from period 1 to period T. For example, inflation from 1947 through 2013 could be indicated with this notation by denoting x_t as inflation, defining period 1 as 1947, and period T as 2013. Obviously many common economic variables qualify across a wide range of fields. Examples may include GDP and inflation in macroeconomics, asset prices in finance (or macroeconomics), and mercury content in a certain river in environmental economics.

A few general points on times series analysis:

1. This chronological component creates a type of relationship among any two observations that does not exist in a cross sectional sample. GDP in 2012, for example, has a different relationship with GDP in 2011 than with GDP in 1942. This temporal relationship creates both complications and opportunities econometrically that are a focus of this course. The goal of an econometrician is to use all available information efficiently. This course is about using the temporal relationships that exists across observations efficiently.

2. A time series is just one realization of a data generating process. Time series analyzes a world with randomness. Therefore any observed time series is simply one of an infinite # of possible realizations that could have occurred.

3. Most empirical macroeconomics involves time series econometrics.

Many of the most pressing questions in short-run macroeconomics concern the relationship among variables such as output, interest rates, taxes, etc. These variables are typically measured as a time series (although it is possible that the data consist only of interest rates, for example,

[1] These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

measured in different countries at the same point in time). As a result, most efforts to use econometrics to answer questions in short run macroeconomics involve time-series.

Long-run macroeconomics is more likely to rely on variation across countries only. Most applied econometrics in this area does, however, also employ time series techniques.

4. Time series is not just empirical macroeconomics.

Although macroeconomics is especially related to time series, many other fields within economics and other disciplines use these techniques as well. Empirical finance, a subfield of economics, almost always relies on time series such as asset prices, profits, etc. Many fields of microeconomics also utilize variables that vary over time. Other disciplines such as biology and engineering frequently employ time series as well.

5. Panel data varies cross sectionally and by time.

Suppose, for example, that we wish to understand the relationship among GDP growth and interest rates. Econometricians looking for data may choose to rely on the historic (time series) variation that has occurred with these variables in U.S. history. But they may also choose to rely on variation across countries. A *panel* is a dataset that varies in both dimensions.

In this example, we might collect annual data going back 50 years for 10 different countries. We might denote this panel's dimensions as $T = 50$ and $N = 10$, for a total of $TxN = 500$ observations.

Panels are vary common in almost all fields of economics. Many (most?) senior theses here at Bates use them. They will be a major focus of this course.

Lag Operators

This course will at times pause to develop mathematical techniques needed to study the material. Our first example is the *lag operator*, denoted L. These are best seen through examples. Suppose that we have some observation x_t. Then:

$$LX_t = x_{t-1} \tag{1}$$

In words, the lag operator moves the variable back one period.

$$L(Lx_t) = L^2 x_t = x_{t-2} \tag{2}$$

More generally,

$$L^k x_t = X_{t-k} \tag{3}$$

Consider the following example:

$$x_t = ax_{t-1} + bx_{t-2} + cx_{t-3} + dx_{t-4} \tag{4}$$

This may be re-written using lag operators as:

$$x_t = (aL + bL^2 + cL^3 + dL^4)x_t \tag{5}$$

Ergodicity and Stationarity

In your earlier courses, you studied *moments*. The first moment of a time series (or any other variable), is the mean. The sample mean for a variable is simply:

$$E[x_t] = \frac{1}{T}\sum_{t=1}^{T} x_t \tag{6}$$

Note that there is a distinction between the sample mean and the true mean. The sample is just the sum of the observed variable over the time series divided by the sample size. But this need not equal the true mean that is yielded by the data generating process which is usually random. This is why (6) includes an expectations operator, the sample mean is just our best guess of the true mean.

The second moment of a time series is the variance. This is given by:

$$V[x_t] = E[(x_t - E[x_t])^2] \tag{7}$$

A constant, by definition, always equals its mean and its variance is thus zero. Once again, there is a distinction between the sample variance and true variance.

The third moment is skewness and the fourth is kurtosis. There are an infinite number of moments. We will almost always focus on the first two.

The concept of *ergodicity* refers to whether or not, if given an infinite sample size (T), a variable's sample moments converge to its true moments. For example, a time series is mean ergodic if and only if:

$$T \overset{lim}{\rightarrow} \infty\ E_t\left[\frac{1}{T}\sum_{t=1}^{T} x_t - u\right] = 0 \tag{8}$$

where u is the true mean. In words, mean ergodicity implies that as the time series gets longer and longer, the sample mean converges to the true mean.

Variance ergodicity implies that as the sample size approaches infinity, the sample variance converges to the true variance:

$$T \overset{lim}{\rightarrow} \infty\ E_t\left[\frac{1}{T}\sum_{t=1}^{T}(x_t - E[x_t])^2 - \sigma_x^2\right] = 0 \tag{9}$$

where σ_x^2 is the true variance. A totally ergodic process will not only satisfy (8) and (9) will also satisfy related properties for all higher moments as well.

We cannot observe the true moments of a variable. We thus have no way of testing for ergodicity. Instead, we typically assume it. If, however, the underlying process is non-ergodic, then the subsequent results will be biased.

We now consider the related concept of *stationarity*. An ergodic process must be stationary. It is, however, possible for a process to be stationary but not ergodic. Formally, stationarity implies that a variable's true distribution be independent of time. Note that this applies to the true moments, not the sample moments. The latter will typically be sensitive to when they are measured.

Suppose, for example, that we observe two samples of a time series. The early sample has a low mean while the later sample has a high mean. One possibility is that the means differ because of the time difference. This would be the case, for example, if the U.S. population were the time series. In this case, we say that the time series is non-stationary. It is also possible, however, that the means differ due to the inherent randomness of the data generating process. If this has nothing to do with time, then we say that the variable is stationary, at least as far as the mean is concerned.

Stationarity is crucial. Non-stationarity violates the Gauss-Markov conditions and renders most of the techniques developed in this class invalid. Fortunately, we will learn to test for and handle non-stationarity. Were we to fail to do this, and simply plow ahead without making sure our econometric specification deals with possible non-stationarity, we would risk generating useless results.

A truly stationary variable has all its moments be time-independent. We, however, consider a version known as *weak stationarity.* This implies that the variables mean, variance, and the covariance between any two lags of the variables be time-independent. These covariances may be written as:

$$Cov[x_t, x_s] = E[(x_t - E[x_t])(x_s - E[x_s])] \tag{10}$$

Fortunately, weak stationarity is sufficient for most of this course's applications.

Stationarity is not a concept that comes easily to most students. We will, however, see it throughout the course. Consider a simple example:

$$x_t = \alpha t + u_t \tag{11}$$

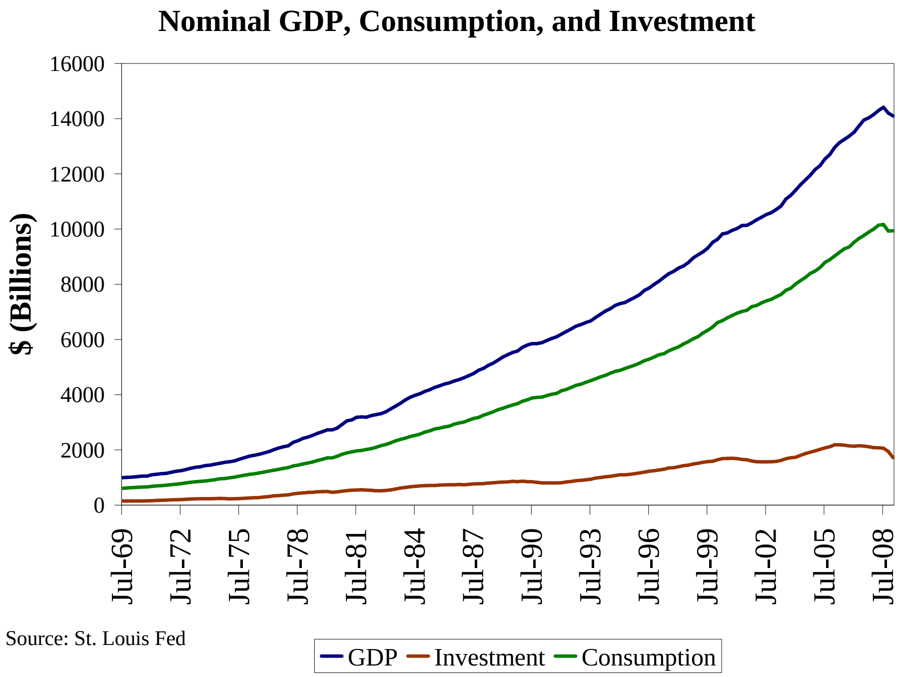
where u_t is a random variable with mean equal to $\bar{u}$. If u_t is unrelated to any values of u in other periods, then it is said to be *white noise.*

Suppose that $\alpha = 1$ and $\bar{u} = 0$. If we take the period $t \in [0, 10]$, then the true mean is 5. If we take it for $t \in [10, 20]$, however, then it is 15. Clearly, the mean depends on time. The process is thus non-stationary.

Suppose, however, that $\alpha = 0$. In this case, the true mean is zero for any period. The time series is thus mean-stationary. If the variance is also time independent (which we have said nothing about), then it qualifies as weakly stationary.

Note that in both cases, the sample means for any two (finite) time periods will almost surely differ. Suppose you flip a coin 100 times and heads comes up 53 times. Were you to repeat this exercise, you would probably not get 53 heads. But this is because of the randomness of the process. The true distribution would be the same. Keep in mind that stationarity refers to the true moments.

Stationarity implies mean-reversion. It is therefore often possible to eyeball non-stationary processes. Consider the following data on nominal U.S. GDP, consumption, and investment:



Clearly, each of these time series are growing over time with only occasional decreases (recessions). There is no tendency to revert to a time-invariant mean. When we learn to test for stationarity, we will almost surely conclude that all three are not stationary.