The Life-Cycle and Real Business Cycle Models[1]

These notes present two theories. The first, the Life-Cycle Model (LCM) represents the profession's best efforts to explain consumption. The second, the Real Business Cycle Model (RBC), appends the LCM to result in a model of business cycles. Depending on where we are in the class, we may skip the RBC model.

LCM

The LCM is a microfounded model of consumption, it solves for consumption as a choice made by rational utility-maximizing households. It is not a model of business cycles. It represents macroeconomics' best attempt to explain consumption and is a part of most modern theories of the business cycle. Although its exact form varies, the following presentation is generally representative of the theory. We assume the following:

1. For simplicity, the economy consists of one large household.

2. For simplicity, the interest rate always equals zero, and the household does not discount the future. We will relax these assumptions later.

3. The household starts with assets equal to A_1.

4. The household pays lump sum taxes each period equal to T_t.

5. The household's utility function in period t is $U(C_t) = ln(C_t)$. Recall that $\frac{\partial ln(C_t)}{\partial C_t} = \frac{1}{C_t}$.

6. The household lives for N periods.

7. For simplicity, the household knows the future values of all variables with certainty.

The next step is to write out the household's budget constraint. It requires that total consumption must be equal to assets plus future income.

$$\sum_{t=1}^{N} C_t = A_1 + \sum_{t=1}^{N} (Y_t - T_t) \tag{1}$$

The household's utility maximizing problem is then:

[1] These are undergraduate lecture notes. They do not represent academic work. Expect typos, sloppy formatting, and occasional (possibly stupefying) errors.

$$\max_{C_t} \quad \sum_{t=1}^{N} ln(C_t) \tag{2}$$

We will solve this problem using a simple proof. The method is known as a proof by contradiction. We start by assuming the opposite of what we want to prove.

Suppose that for any two periods, s and w, that $C_s > C_w$ is optimal (utility maximizing). Now suppose that the household transfers a very small amount, e of consumption from period s to period w. The marginal utility gain from this exchange is $\frac{e}{C_w}$. The marginal utility loss is $\frac{e}{C_s}$. By assumption, the former is larger than the latter. This change thus increases lifetime utility.

It follows that any allocation where $C_s > C_w$ cannot be optimal. It must thus be the case that $C_s = C_w$.

It further follows that consumption in each period is the sum of assets, income, and future income divided by the number of periods:

$$C_t = \frac{1}{N}\left[A_1 + \sum_{t=1}^{N}(Y_t - T_t)\right] \tag{3}$$

Note that any utility function where marginal utility is decreasing in consumption yields this same result. (Formally, such a utility function is concave in that its second derivative is negative.)

For the more mathematically inclined, we can also solve this problem using a Lagrangian (we will not be doing this in class):

$$L = \sum_{t=1}^{N} ln(C_t) + \lambda\left(A_1 + \sum_{t=1}^{N}(Y_t - T_t) - \sum_{t=1}^{N} C_t\right) \tag{4}$$

Differentiating with respect to C_t yields:

$$\frac{1}{C_t} = \lambda \tag{5}$$

For the marginal utility of consumption to be equal for all t, so must the value of consumption. It follows that $C_1 = C_2 = C_3....$, and (3) must again hold.

Equation (3) has several important implications:

1. Consumption depends on lifetime wealth which includes not only current disposable income, but current assets and future disposable income as well. The Life-cycle model is thus a much richer model of consumption than something like a simple Keynesian consumption function where C_t depends only on current disposable income. More generally, consumption in the LCM depends on expected discounted future income. But we have simplified the model by assuming no uncertainty and zero interest rates for now.

2. Households smooth their consumption, in this case perfectly so. Because the utility function exhibits decreasing marginal utility (formally, it is concave), households prefer 5 units of consumption each period instead of 0 then 10 or vice-versa.

3. Suppose that a household exhibits a one-time, one-unit increase in its disposable income. According to the model, the household will increase its consumption by only $\frac{1}{N}$ units. It does not matter if the income increase occurs in the present or in the future. This result occurs because households wish to smooth the benefits of this income increase over time. They thus increase their savings in period t.

4. Suppose, however, that the household receives a one-unit permanent increase in its income. In this case, consumption will increase by a full unit. Permanent changes to income thus have larger effects than temporary increases. For this reason the LCM is also known as the *permanent income hypothesis*.

5. The previous result has important implications for tax policy. Suppose, for example, that Congress is considering a tax cut. A temporary tax cut will likely have only a small effect on consumption because households will save most of the tax cut in order to enjoy more future consumption. A permanent tax cut, however, will have a larger effect on spurring consumption.

6. Suppose that the government also has a budget constraint:

$$\sum_{t=1}^{Z} G_t + D_1 = \sum_{t=1}^{Z} T_t \tag{6}$$

where D_1 is the initial level of governmental debt. In words, the government must pay down its debt by the end of period Z. Suppose that $Z < N$, and the government cuts taxes in period t without changing the path of current and future government spending. In this case, the tax cut will have no effect because the household knows that the government will have to raise taxes by the same amount before period T. The tax change has no effect on lifetime wealth and hence no effect on current consumption. This result is known as *Ricardian Equivalence.*

Suppose, however, that $Z < N$. In this case, the government need not pay off its debt before the current set of households dies. In this case, the tax cut still may (depending on the government's specific plans) increase C_t. Because the government's horizon is longer than that of the households, this type of intergenerational transfer is possible.

7. Suppose that the government increases G_t in order to boost output. An analysis similar to #6 applies. If increased G_t leads to higher taxes within the households' lifetime, then the effects of the fiscal stimulus will be reduced.

The Euler Equation

So far, we have examined a simplified version of the model. We now expand the model in three ways. First, households do not know the future but must instead rely on expectations. The term $E_t[X_{t+1}]$ indicates the expectation, formed in period t, of X in period $t+1$. E is an operator that indicates an expectation. Second, the interest rate, r_t is no longer assumed to equal zero. Third, we assume that households discount the future so that one unit of utility in period $t+1$ is worth β units in period t. We assume that β is between zero and one. The households' objective function is now:

$$\max_{C_t} \quad E_1[\beta^{t-1}\sum_{t=1}^{N} ln(C_t)] \tag{7}$$

Likewise, the households budget constraint must now be discounted using r_t. At any point in time, the budget constraint takes the following form:

$$C_1+E_1[\frac{C_2}{1+r_1}]+E_1[\frac{C_3}{(1+r_1)(1+r_2)}]+.. = A_1+(Y_1-T_1)+E_1[\frac{Y_2-T_2}{1+r_1}]+E_1[\frac{Y_3-T_3}{(1+r_1)(1+r_2)}]+.. \tag{8}$$

We will solve this version of the model with a thought experiment in several steps. To make it easier, assume that there are no taxes.

1. Suppose that a household increases its current consumption by one (very small) unit. Doing so increases its utility by $\frac{1}{C_t}$ in period t.

2. By increasing its consumption in period t, the households wealth in period $t+1$ is reduced by $(1 + r_t)$ units.

3. Suppose that the household then expects to reduce its consumption in period $t + 1$ by $(1 + r_t)$ units. Doing so will leave its wealth in period $t + 2$ and beyond unchanged from the baseline.

4. The expected lost utility from #3 is $\frac{1+r_t}{C_{t+1}}$. Because the household discounts, however, this must be multiplied by β.

5. If the benefits of this exercise exceed its costs, then the household, by definition, cannot me maximizing its lifetime utility. Likewise, if the costs exceed the benefits then the household could increase its utility be reducing C_t. So it must be true that the costs equal the benefits. This yields the *Euler Equation*:

$$\frac{1}{C_t} = E_t \left[\frac{\beta(1 + r_t)}{C_{t+1}} \right] \tag{9}$$

The Euler (it is pronounced like "Oiler") Equation is a representation of the LCM's equilibrium. It predicts that households try to smooth their consumption. But changes in the interest rate may cause them to no longer perfectly do. If r_t is very high, for example, then households have an additional incentive to save. They will thus accept less consumption in t for higher expected consumption in $t + 1$. Likewise low interest rates reduce the incentive to save and increase C_t.

Euler equations are a hallmark of modern macro. They represent the LCM and are a part of most business cycle models. Note that changing the interest rate can affect current consumption. This will be crucial when the class gets to monetary policy.

The Real Business Cycle Model

The LCM, through the Euler Equation, can be appended to obtain the Real Business Cycle Model (RBC). There are three additions needed. First, in addition to choosing consumption, households also choose their labor supply. To do this, we extend the utility function so that:

$$U(C_t, L_t) = ln(C_t) - \nu ln(L_t) \tag{10}$$

This utility function imposes that households obtain both utility from consumption and disutility from supplying labor. The parameter ν represented the weight placed on labor/leisure in the utility function. Optimization requires that the marginal benefit of consumption equal the marginal disutility of labor:

$$\frac{w_t}{C_t} = \frac{\nu}{L_t} \tag{11}$$

The second addition is to introduce random productivity shocks into the model. Total factor productivity, A_t is hit with random and exogenous shocks that temporarily move it away from its mean value. This assumption is where the "real" in "real business cycle" comes from. The business cycle will not be influenced by nominal shocks to things like money, but instead by events that effect the productivity of a given amount of inputs.

The third addition is to introduce capital into the model. We can then assume a now familiar production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{12}$$

If the labor market is competitive, then the wage equals the marginal product of labor. This allows us to rewrite (11) as:

$$\frac{(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}}{C_t} = \frac{\nu}{L_t} \tag{13}$$

By assuming a competitive market for capital, the interest rate equals the marginal product of capital less depreciation:

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - d \tag{14}$$

Inserting (14) into (9) allows us to rewrite the Euler Equation:

$$\frac{1}{C_t} = E_t \left[\frac{\beta(1 + \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - d)}{C_{t+1}} \right] \tag{15}$$

The final equation in the model is a capital accumulation equation:

$$K_{t+1} = (1-d)K_t + Y_t - C_t \tag{16}$$

Formally solving the model is beyond the scope of the class. The effects of a negative productivity shock are, however, intuitive:

1. All else equal, lower productivity reduces both output and the marginal product of capital.

2. From (12) interest rates fall.

3. From (13), as the marginal product of labor falls, so does wages. Households thus rationally choose to supply less labor.

4. From (15), households attempt to smooth out the adverse effects of the productivity shock by reducing their saving.

5. From (16) the economy enters the next period with less capital than it otherwise would have had. The effects of the shock are thus propagated and may outlast the period of reduced productivity.

In the RBC model, short run fluctuations are rational responses to productivity shocks. recessions, for example, are efficient. Employment declines because the return on labor decreases so households rationally choose to substitute toward more leisure.

Testing the RBC Model

To judge the RBC model, we compare its predictions with those from the actual U.S. data. To simulate the model, we do the following:

1. We first calibrate the model. This entails assigning general parameters specific numerical values.For example, we set depreciation, d, equal to 2.5% per quarter because this roughly matches what we observe on the microeconomic level.

2. We then generate a series of productivity shock.s One issue is to choose the variance of A_t. There are a few ways to do this. Here, however, we choose this variance so that the simulated standard deviation of output matches the value observed in the data.

3. After generating these shocks, we use the equations of the model to solve for the model's endogenous variables.

One way to judge the model is to compare the simulated versus actual standard deviations of the model's endogenous variables:

Note that the volatilities of output match up. This is not an advantage of the model, we choose the standard deviation of A_t so that this would happen. On the whole, the RBC model

Table 1: Standard deviation of key variables: $\frac{\hat{\beta}}{\beta} = 0.75$

	Y_t	C_t	I_t	L_t	w_t
U.S. Data	1.81	1.35	5.30	1.79	0.68
RBC	1.81	0.79	5.33	0.87	0.98

does fairly well. It does predict that consumption and employment are more stable than it actually is and that the wage is more volatile than it actually is.

Another way to test the model is to look at variable's correlations with output.

Table 2: Correlation with Y_t

	C_t	I_t	L_t	w_t
U.S. Data	0.88	0.80	0.88	0.12
RBC	0.94	0.99	0.97	0.98

Here the RBC model does well with one exception. It predicts that the wage rate is strongly correlated with output (procyclical) when in reality this correlation is weak.

Collectively, I view these tests as showing that the RBC model makes for a good first pass at modeling business cycles. There are, however, a few additional problems that keep us from taking it too seriously. First, it does not include some other factors, such as money and financial market imperfections, that are empirically supported and important to understanding actual business cycles. Second, this version of the RBC model does not model involuntary unemployment. In the model, all unemployment is a rational and efficient response to exogenous productivity shocks. The remedy these problems, we next turn to the New Keynesian model.